Submetacompactness in countable products (Problems and applications in General and Geometric Topology)

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1 Introduction

A space X is said to be *metacompact* if every open cover of X has a point finite open refinement and X is said to be *submetacompact* if for every open cover $\mathcal{U}$ of X, there is a sequence $(\mathcal{V}_n)_{n\in\omega}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$, there is an $n \in \omega$ with $\mathrm{Ord}(x, \mathcal{V}_n) < \omega$. Here, for $x \in X$ and $n \in \omega$, let $\mathcal{V}_{n_x} = \{V \in \mathcal{V}_n : x \in V\}$ and $\mathrm{Ord}(x, \mathcal{V}_n) = |\mathcal{V}_{n_x}|$. We call this sequence $(\mathcal{V}_n)_{n\in\omega}$ a *θ-sequence* of open refinemenets of $\mathcal{U}$. Clearly, every paracompact space is metacompact and every metacompact space is submetacompact. It is well known that if X is countably compact and submetacompact, then X is compact.

Since the notion of C-scattered spaces was introduced by R. Telgársky [Te1], C-scattered spaces play the fundamental role in the study of covering properties of products. A space X is said to be *scattered* if every nonempty subset A of X has an isolated point in A, and X is said to be *C-scattered* if for every nonempty closed subset A of X, there is a point $x \in A$ which has a compact neighborhood in A. Scattered spaces and locally compact spaces are C-scattered. R. Telgársky [Te1] proved that if X is a C-scattered paracompact (Lindelöf) space, then $X \times Y$ is paracompact (Lindelöf) for every paracompact (Lindelöf) space Y.

R. Telgársky [Te2] also introduced the notion of $\mathcal{DC}$-like spaces, using topological games. The class of $\mathcal{DC}$-like spaces includes all spaces with a σ-closure-preserving closed cover by compact subsets and all paracompact C-scattered spaces. R. Telgársky proved that if X is a paracompact (Lindelöf) $\mathcal{DC}$-like space, then $X \times Y$ is paracompact (Lindelöf) for every paracompact (Lindelöf) space Y. Furthermore, G. Gruenhage and Y. Yajima [GY] proved that if X is a metacompact (submetacompact) $\mathcal{DC}$-like space, then $X \times Y$ is metacompact (submetacompact) for every metacompact (submetacompact) space Y and that if X is a C-scattered metacompact (submetacompact) space, then $X \times Y$ is metacompact (submetacompact) for every metacompact (submetacompact) space Y. For covering properties of countable products, the author proved the following.

(A) ([T1]) If Y is a perfect paracompact (hereditarily Lindelöf) space and $\{X_n : n \in \omega\}$ is a countable collection of paracompact (Lindelöf) $\mathcal{DC}$-like spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact (Lindelöf).

(B) ([T2, T3]) If $\{X_n : n \in \omega\}$ is a countable collection of metacompact (submetacompact) $\mathcal{DC}$-like spaces, then the product $\prod_{n\in\omega} X_n$ is metacompact (submetacompact).

(C) ([T2]) If $\{X_n : n \in \omega\}$ is a countable collection of C-scattered metacompact spaces, then the product $\prod_{n\in\omega} X_n$ is metacompact.

The author [T3] asked whether the product $\prod_{n\in\omega} X_n$ is submetacompact whenever X_n is a C-scattered submetacompact space for each $n \in \omega$.

Our purpose of this paper is to give an affirmative answer to this problem.

All spaces are assumed to be regular T_1. Let ω denote the set of natural numbers and $|A|$ denote the cardinality of a set A. Undefined terminology can be found in R. Engelking [E].

2 Submetacompactness

Let X be a space. For a closed subset A of X, let

$$A^* = \{x \in A : x \text{ has no compact neighborhood in } A\}.$$

Let $A^{(0)} = A, A^{(\alpha+1)} = (A^{(\alpha)})^*$ and $A^{(\alpha)} = \cap_{\beta<\alpha} A^{(\beta)}$ for a limit ordinal α. Note that every $A^{(\alpha)}$ is a closed subset of X and if A and B are closed subsets of X such that $A \subset B$, then $A^{(\alpha)} \subset B^{(\alpha)}$ for each ordinal α. Furthermore, X is C-scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α. Let X be a C-scattered space and $A \subset X$. Put $\lambda(X) = \inf\{\alpha : X^{(\alpha)} = \emptyset\}$ and $\lambda(A) = \inf\{\alpha : A \cap X^{(\alpha)} = \emptyset\} \leq \lambda(X)$.

It is clear that if A, B are subsets of X such that $A \subset B$, then $\lambda(A) \leq \lambda(B)$. A subset A of X is said to be *topped* if there is an ordinal $\alpha(A)$ such that $A \cap X^{(\alpha(A))}$ is a nonempty compact subset of X and $A \cap X^{(\alpha(A)+1)} = \emptyset$. Thus $\lambda(A) = \alpha(A) + 1$. For each $x \in X$, there is a unique ordinal α such that $x \in X^{(\alpha)} - X^{(\alpha+1)}$, which is denoted by $\text{rank}(x) = \alpha$. There is a neighborhood base $\mathcal{B}_x$ of x in X, consisting of open subsets of X, such that for each $B \in \mathcal{B}_x$, clB is topped in X and $\alpha(clB) = \text{rank}(x)$. If A is a topped subset of X and B is a subsets of A such that $B \cap (A \cap X^{(\alpha(A))}) = B \cap X^{(\alpha(A))} = \emptyset$, then $\lambda(B) \leq \alpha(A) < \lambda(A) = \alpha(A) + 1$.

The following plays the fundamental role in the study of submetacompactness.

LEMMA 2.1. (G. Gruenhage and Y. Yajima [GY]) *There is a filter $\mathcal{F}$ on ω satisfying: For every submetacompact space X and every open cover $\mathcal{U}$ of X, there is a sequence $(\mathcal{V}_n)_{n\in\omega}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$,*

$$\{n \in \omega : Ord(x, \mathcal{V}_n) < \omega\} \in \mathcal{F}.$$

By Lemma 2.1, let $\mathcal{F}^{n+1}$ denote the filter on ω^{n+1} generated by sets of the form

$$\textstyle\prod_{i\leq n} F_i, \text{ where } F_i \in \mathcal{F} \text{ for each } i \leq n.$$

Put

$$\Phi_n = \textstyle\prod_{i\leq n} \omega^{i+1} \text{ for each } n \in \omega \text{ and } \Phi = \cup\{\Phi_n : n \in \omega\}.$$

For $\mu = (\tau_0, \tau_1, \cdots, \tau_n) \in \Phi_n, n \in \omega$ with $n \geq 1$, let $\mu_- = (\tau_0, \tau_1, \cdots, \tau_{n-1}) \in \Phi_{n-1}$. If $\tau \in \Phi_0$, let $\tau_- = \emptyset$. For each $\tau \in \omega^{n+2}$, let $\mu \oplus \tau = (\tau_0, \tau_1, \cdots, \tau_n, \tau) \in \Phi_{n+1}$. Let $\mathcal{U}, \mathcal{V}$ be collections of subsets of a space X. Put $\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

THEOREM 2.2. *If $\{X_n : n \in \omega\}$ is a countable collection of C-scattered submetacompact spaces, then the product $\prod_{n\in\omega} X_n$ is submetacompact.*

PROOF. We may assume the following (cf. [A1, Theorem]):

(1) X is a C-scattered submetacompact space and for each $n \in \omega$, $X_n = X$,

(2) X is topped and there is a point $a \in X$ such that $X^{(\alpha(X))} = \{a\}$.

We shall show that X^ω is submetacompact. Let $\mathcal{B}$ be the base of X^ω, consisting of all basic open subsets of X^ω, that is $B = \prod_{n\in\omega} B_n \in \mathcal{B}$ if there is an $n \in \omega$ such that for $i < n, B_i$ is an open subset of X and for $i \geq n, B_i = X$. Let

$$n(B) = \inf\{i : B_j = X \text{ for } j \geq i\}.$$

We call $n(B)$ the *length* of B. Let $\mathcal{O}$ be an open cover of X^ω, which is closed under finite unions and $\mathcal{O}' = \{B \in \mathcal{B} : B \subset O \text{ for some } O \in \mathcal{O}\}$.

Take a $B = \prod_{i\in\omega} B_i \in \mathcal{B}$ and let $\mathcal{N}(B) = \{i < n(B) : clB_i \text{ is topped in } X\}$. Take an $i < n(B)$ with $i \notin \mathcal{N}(B)$. In case ot that $\lambda(clB_i)$ is an isolated ordinal. Then there is an ordinal γ such that $\lambda(clB_i) = \gamma + 1$ and $clB_i \cap X^{(\gamma)}$ is nonempty and locally compact. For each $x \in clB_i \cap X^{(\gamma)}$, there is an open neighborhood B_x of x in X such that clB_x is topped in X, $cl(B_x \cap X^{(\gamma)})$ is compact and $\alpha(clB_x) = \text{rank}(x)$. For each $x \in clB_i - X^{(\gamma)}$, take an open neighborhood B_x of x in X such that clB_x is topped in $X, clB_x \cap (clB_i \cap X^{(\gamma)}) = \emptyset$ and $\alpha(clB_x) = \text{rank}(x)$. Then every $clB_i \cap clB_x$ is topped in X and $\alpha(clB_i \cap clB_x) = \alpha(clB_x)$. Next, let $i = n(B)$. Since $X^{(\alpha(X))} = \{a\}$, take a proper open neighborhood B_a of a in X, and for each $x \in X - \{a\}$, take an open neighborhood B_x of x in X such that $a \notin clB_x, clB_x$ is topped in X and $\alpha(clB_x) = \text{rank}(x)$. In case of that $\lambda(clB_i)$ is a limit ordinal. For each $x \in clB_i$, there is an open neighborhood B_x of x in X such that clB_x is topped in X and $\alpha(clB_x) = \text{rank}(x)$. Since $\mathcal{B}_i(B) = \{B_x : x \in clB_i\}$ is an open cover of clB_i and X is submetacompact, there is a θ-sequence $(\mathcal{V}^j_{B,i})_{j\in\omega}$ of open (in X) refinements of $\mathcal{B}_i(B), \mathcal{V}^j_{B,i} = \{V_\xi : \xi \in \Xi^j_{B,i}\}, j \in \omega$, such that for each $j \in \omega, \cup\mathcal{V}^j_{B,i} = B_i$ and for each $x \in B_i, \{j \in \omega : Ord(x, \mathcal{V}^j_{B,i}) < \omega\} \in \mathcal{F}$, where $\mathcal{F}$ is the filter on ω described in Lemma 2.1. For each $j \in \omega$ and $\xi \in \Xi^j_{B,i}$, take an $x(\xi) \in clB_i$ such that $V_\xi \subset B_{x(\xi)}$. For each $i \in \mathcal{N}(B)$ and $j \in \omega$, let $\Xi^j_{B,i} = \{\xi^j_{B,i}\}$ and $\mathcal{V}^j_{B,i} = \{V_{\xi^j_{B,i}}\} = \{B_i\}$. For each $\eta = (j_0, j_1, \cdots, j_{n(B)}) \in \omega^{n(B)+1}$, put $\Xi_{B,\eta} = \prod_{i\leq n(B)} \Xi^{j_i}_{B,i}$. For each $\xi = (\xi(i)) \in \Xi_{B,\eta}$, let $V(\xi) = \prod_{i\leq n(B)} V_{\xi(i)} \times X \times \cdots$ and $\mathcal{V}_\eta(B) = \{V(\xi) : \xi \in \Xi_{B,\eta}\}$. Then every $\mathcal{V}_\eta(B)$ is an open cover of B. Take a $\xi = (\xi(i)) \in \Xi_{B,\eta}$ and let $\mathcal{M}(\xi) = \{i \leq n(B) : clV_{\xi(i)} \text{ is topped in X}\}$. Then $\mathcal{N}(B) \subset \mathcal{M}(\xi)$. Put $K(\xi) = \prod_{i\in\mathcal{M}(\xi)} (clV_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))}) \times \prod_{i\leq n(B), i\notin\mathcal{M}(\xi)} V_{\xi(i)} \times \{a\} \times \cdots = \prod_{i\in\omega} K_{\xi,i}$ and $\mathcal{K}(B,\eta) = \{K(\xi) : \xi \in \Xi_{B,\eta}\}$. We consider the following condition for $K(\xi)$.

(*) There is an open set $B' \in \mathcal{O}'$ such that $K(\xi) \subset B'$.

If $K(\xi)$ satisfies (*), define $n(\xi) = \inf\{n(O) : K(\xi) \subset O \text{ and } O \in \mathcal{O}'\}$. Put $r(\xi) = \max\{n(B), n(\xi)\}$. Then there is an $O(\xi) = \prod_{i\in\omega} O_{\xi,i} \in \mathcal{O}'$ such that:

(3) $K(\xi) \subset O(\xi)$,

(4) $n(\xi) = n(O(\xi))$.

Take an $H(\xi) = \prod_{i\in\omega} H_{\xi,i} \in \mathcal{O}'$ such that:

(5) (a) $\prod_{i<n(\xi)} H_{\xi,i} \times X \times \cdots \subset O(\xi)$,

(b) for i with $n(\xi) \leq i \leq n(B)$ or $i \leq n(B)$ with $i \notin \mathcal{M}(\xi)$, let $H_{\xi,i} = O_{\xi,i}$,

(c) for $i < n(\xi)$ with $i \in \mathcal{M}(\xi)$, let $H_{\xi,i}$ be an open subset of X such that $K_{\xi,i} = clV_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))} \subset H_{\xi,i} \subset clH_{\xi,i} \subset O_{\xi,i}$,

(d) for i with $n(B) < i < n(\xi)$, let $H_{\xi,i}$ be an open subset of X such that $K_{\xi,i} = \{a\} \in H_{\xi,i} \subset clH_{\xi,i} \subset O_{\xi,i}$,

(e) if $r(\xi) = n(B)$, let $H_{\xi,i} = X$ for each $i > n(B)$, and if $r(\xi) = n(\xi) > n(B)$, let $H_{\xi,i} = X$ for $i \geq n(\xi)$.

Then we have $K(\xi) \subset H(\xi)$. Let $\mathcal{P}(B) = \{P : P \subset \{0, 1, \cdots, n(B)\}\}$ and $P \in \mathcal{P}(B)$. Define

$$G(\xi) = \prod_{i\in\omega} G_{\xi,i} \text{ and } B(\xi, P) = \prod_{i\in\omega} B_{\xi,P,i}$$

as follows:

(6) (a) In case of that $r(\xi) = n(B)$. For each $i \leq n(B)$, let $G_{\xi,i} = V_{\xi(i)} \cap O_{\xi,i}$ and for each $i > n(B)$, let $G_{\xi,i} = X$.

(b) In case of that $r(\xi) = n(\xi) > n(B)$. For each $i \in \omega$, let $G_{\xi,i} = \emptyset$.

(c) In either case, for each $i \leq n(B)$, if $i \in P$, let $B_{\xi,P,i} = V_{\xi(i)} - clH_{\xi,i}$ and if $i \notin P$, let $B_{\xi,P,i} = V_{\xi(i)} \cap O_{\xi,i}$. For each $i > n(B)$, let $B_{\xi,P,i} = X$.

Clearly, if $r(\xi) = n(B)$, then $B(\xi, \emptyset) = G(\xi)$. Notice that for each $i \in \omega$, $B_{\xi,P,i} \subset B_i$ and if $B(\xi, P) \neq \emptyset$, then $n(B(\xi, P)) = n(B) + 1$. Let $i \leq n(B)$. If $i \in P$ and $i \notin \mathcal{M}(\xi)$, then $B_{\xi,P,i} = \emptyset$.

Let

$$\mathcal{B}_{\eta,\xi}(B) = \{B(\xi, P) : P \in \mathcal{P}(B) - \{\emptyset\}\} \text{ if } r(\xi) = n(B),$$
$$\mathcal{B}_{\eta,\xi}(B) = \{B(\xi, P) : P \in \mathcal{P}(B)\} \text{ if } r(\xi) = n(\xi) > n(B),$$

CLAIM 1. *Let $K(\xi)$ satisfies the condition (*), $P \in \mathcal{P}(B)$ and $B(\xi, P) \in \mathcal{B}_{\eta,\xi}(B)$ with $B(\xi, P) \neq \emptyset$. If $r(\xi) = n(B)$, then there is an $i < n(\xi)$ with $i \in P$.*

Now, assume that $K(\xi)$ does not satisfy the condition (*). Let $G(\xi) = \emptyset$. Take a $P \in \mathcal{P}(B)$ and define $B(\xi, P)$ as follows: If $P = \emptyset$, let $B(\xi, P) = V(\xi)$. If $P \neq \emptyset$, let $B(\xi, P) = \emptyset$. Put $\mathcal{B}_{\eta,\xi}(B) = \{B(\xi, P) : P \in \mathcal{P}(B)\} = \{V(\xi)\}$.

Then, in each case, we have $V(\xi) = G(\xi) \cup (\cup\mathcal{B}_{\eta,\xi}(B))$.

CLAIM 2. *Let $i \leq n(B), \xi = (\xi(i)) \in \Xi_{B,\eta}, K(\xi) = \prod_{t\in\omega} K_{\xi,t}, P \in \mathcal{P}(B)$ and $B(\xi, P) = \prod_{t\in\omega} B_{\xi,P,t}$ with $B_{\xi,P,i} \neq \emptyset$.*

(a) *If $i \in P$, then $K(\xi)$ satisfies (*), $i \in \mathcal{M}(\xi)$ and $\lambda(clB_{\xi,P,i}) < \lambda(clB_i)$.*

(b) *Let $i \notin P$.*

(i) *If $i \in \mathcal{M}(\xi)$, then $clB_{\xi,P,i}$ is topped in X such that $\lambda(clB_{\xi,P,i}) = \lambda(clV_{\xi(i)})$ and $K_{\xi,i} = clV_{\xi(i)} \cap X^{(\alpha(clV_{\xi(i)}))} = clB_{\xi,P,i} \cap X^{(\alpha(clB_{\xi,P,i}))}$. Furthermore, if $i \in \mathcal{N}(B)$, then $clB_{\xi,P,i}$ is topped in X such that $\lambda(clB_{\xi,P,i}) = \lambda(clB_i)$ and $K_{\xi,i} = clB_i \cap X^{(\alpha(clB_i))} = clB_{\xi,P,i} \cap X^{(\alpha(clB_{\xi,P,i}))}$,*

(ii) *If $i \notin \mathcal{M}(\xi)$, then $\lambda(clB_{\xi,P,i}) < \lambda(clB_i)$.*

For each $\eta \in \omega^{n(B)+1}$, put

$$\mathcal{G}_\eta(B) = \{G_\xi : \xi \in \Xi_{B,\eta}\} \text{ and}$$
$$\mathcal{B}_\eta(B) = \cup\{\mathcal{B}_{\eta,\xi}(B) : \xi \in \Xi_{B,\eta}\}.$$

Then we have

(7) (a) every element of $\mathcal{G}_\eta(B)$ is contained in some member of $\mathcal{O}'$,

(b) $\mathcal{G}_\eta(B) \cup \mathcal{B}_\eta(B)$ is a cover of B,

(c) the length of nonempty element of $\mathcal{B}_\eta(B)$ is $n(B)+1$.

(8) For each $x \in B, \{\eta \in \omega^{n(B)+1} : \mathrm{Ord}(x, \mathcal{V}_\eta) < \omega\} \in \mathcal{F}^{n(B)+1}$.

(9) For each $x \in B, \{\eta \in \omega^{n(B)+1} : \mathrm{Ord}(x, \mathcal{G}_\eta(B) \cup \mathcal{B}_\eta(B)) < \omega\} \in \mathcal{F}^{n(B)+1}$.

For each $m \in \omega$ and $\tau \in \Phi_m$, let us define $\mathcal{G}_\tau$ and $\mathcal{B}_\tau$ of elements of $\mathcal{B}$. For each $m \in \Phi_0 = \omega$, let $\mathcal{G}_m = \mathcal{G}_m(X^\omega)$ and $\mathcal{B}_m = \mathcal{B}_m(X^\omega)$. Assume that for $m \in \omega$ and $\mu \in \Phi_m$, we have already obtained $\mathcal{G}_\mu$ and $\mathcal{B}_\mu$. Let $\tau \in \Phi_{m+1}$ and $\tau = \mu \oplus \eta$, where $\mu = \tau_- \in \Phi_m$ and $\eta \in \omega^{m+2}$. Let $B \in \mathcal{B}_\mu$. If $B \neq \emptyset$, then we denote $\mathcal{G}_\eta(B)$ and $\mathcal{B}_\eta(B)$ by $\mathcal{G}_\tau(B)$ and $\mathcal{B}_\tau(B)$ respectively. If $B = \emptyset$, let $\mathcal{G}_\tau(B) = \mathcal{B}_\tau(B) = \{\emptyset\}$. Let $\mathcal{G}_\tau = \mathcal{G}_\mu \cup (\cup\{\mathcal{G}_\tau(B) : B \in \mathcal{B}_\mu\})$ and $\mathcal{B}_\tau = \cup\{\mathcal{B}_\tau(B) : B \in \mathcal{B}_\mu\}$. Then every nonempty element of $\mathcal{B}_\tau$ has the length $m+2$.

CLAIM 3. $\{\mathcal{G}_\tau \cup (\mathcal{B}_\tau \wedge \mathcal{O}') : \tau \in \Phi\}$ *is a θ-sequence of open refinements of $\mathcal{O}'$.*

PROOF OF CLAIM 3. By (7) (a) and induction, for each $\tau \in \Phi, \mathcal{G}_\tau \cup (\mathcal{B}_\tau \wedge \mathcal{O}')$ is an open refinement of $\mathcal{O}'$. Take an $x = (x_i) \in X^\omega$. By (9), take a $\tau(0) = m(0) \in \omega$ such that $Ord(x, \mathcal{G}_{\tau(0)} \cup \mathcal{B}_{\tau(0)}) < \omega$. Then $\tau(0) \in \Phi_0$. If $\mathcal{B}_{\tau(0)_x} = \emptyset$, then we are done. So, assume tha $\mathcal{B}_{\tau(0)_x} \neq \emptyset$. By (7) (c), every nonempty element of $\mathcal{B}_{\tau(0)}$ has the length 1. By (9) again, we can take an $\eta(1) \in \omega^2$ such that

$$\eta(1) \in \cap\{\{\eta \in \omega^2 : Ord(x, \mathcal{G}_\eta(B) \cup \mathcal{B}_\eta(B)) < \omega\} : x \in B \in \mathcal{B}_{\tau(0)}\} \in \mathcal{F}^2.$$

Let $\tau(1) = (\eta(0), \eta(1)) \in \Phi_1$. Then we have $Ord(x, \mathcal{G}_{\tau(1)} \cup \mathcal{B}_{\tau(1)}) < \omega$. Assume also that $\mathcal{B}_{\tau(1)_x} \neq \emptyset$. Continuing in this manner, we can choose a $\tau(t) = (\eta(0), \eta(1), \cdots, \eta(t)) \in \Phi_t$ such that for each $t \in \omega$, $\mathrm{Ord}(x, \mathcal{G}_{\tau(t)} \cup \mathcal{B}_{\tau(t)}) < \omega$ and $\mathcal{B}_{\tau(t)_x} \neq \emptyset$. Since $\mathcal{B}_{\tau(t)_x} \neq \emptyset$ and finite for each $t \in \omega$, it follows from König's lemma (cf. K. Kunen [K]) that there are sequences $\{\xi_t : t \in \omega\}, \{\mathcal{N}(t) : t \in \omega\}, \{\mathcal{M}(t) : t \in \omega\}, \{K(t) : t \in \omega\}, \{P(t) : t \in \omega\}, \{B(t) = B(\xi(t), P(t)) : t \in \omega\}, B(\xi(t), P(t)) = \prod_{i\in\omega} B_{t,i}$ of elements of $\mathcal{B}$ satisfying: for each $t \in \omega$,

(10) (a) $x \in B(t) = \prod_{i\in\omega} B(t)_i \in \mathcal{B}_{\eta(t)}(B(t-1))$ and $n(B(t)) = t+1$, where $B(-1) = X^\omega$,

(b) $\xi_t \in \Xi_{B(t-1),\eta(t)}$,

(c) $\mathcal{N}(t) = \mathcal{N}(B(t-1))$,

(d) $\mathcal{M}(t) = \mathcal{M}(\xi_t)$,

(e) $K(t) = K(\xi_t) = \prod_{i\in\omega} K(t)_i \in \mathcal{K}(B(t-1), \eta(t))$,

(f) $P(t) \in \mathcal{P}(\{0, 1, \cdots, n(B(t-1))\})$

(g) if $K(t)$ satisfies the condition (*) and $r(\xi_t) = n(B(t-1))$, then there is an $i < n(\xi_t)$ with $i \in P(t)$,

(h) If $i \in P(t)$, then $\lambda(clB(t)_i) < \lambda(clB(t-1)_i)$,

(i) For $i \notin P(t)$,

(1) if $i \in \mathcal{M}(t)$, then $K(t)_i \subset clB(t-1)_i, i \in \mathcal{N}(t+1)$ and furthermore, if $i \in \mathcal{N}(t)$, then $K(t)_i = clB(t-1)_i \cap X^{(\alpha(clB(t-1)_i))} = K(t+1)_i = clB(t) \cap X^{(\alpha(clB(t)_i))}$ and hence $\lambda(clB(t-1)_i) = \lambda(clB(t)_i)$,

(2) if $i \notin \mathcal{M}(t)$, then $\lambda(clB(t)_i) < \lambda(clB(t-1)_i)$.

Let $i \in \omega$. By (10)(a), let $t \geq 1$ such that $n(B(t)) > i$. By (10)(h), if $i \in P(m)$ for $m \geq t, \lambda(clB(m)_i) < \lambda(clB(m-1)_i)$. Since there does not exist an infinite decreasing sequence of ordinals, there is an $t_i \geq 1$ such that for each $t \geq t_i, i \notin P(t)$. By (10) (i) (2), there is an m_i such that $m_i \geq t_i$ and for each $t \geq m_i, i \in \mathcal{M}(t)$. Then, by (10) (i) (1), for each $t \geq m_i, clB(t+1)_i$ is topped and $clB(t+1)_i \cap X^{(\alpha(clB(t+1)_i))} = K(t+1)_i = K(m_i+1)_i$. Let $K = \prod_{i\in\omega} K(m_i+1)_i$. Then K is a compact subset of X^ω. Since $\mathcal{O}$ is an open cover of X^ω, which is closed under finite unions, there is an $O = \prod_{i\in\omega} O_i \in \mathcal{O}'$ such that $K \subset O$. By (10) (a), take an $s \geq 1$ such that:

(11) (a) $n(O) \leq n(B(s-1))$,

(b) for each $i < n(O), m_i + 1 \leq s$.

For each $i < n(O)$, by (11) (b), $K(s)_i = K(m_i+1)_i \subset O_i$. Then $K(s) \subset O$ and hence, $K(s)$ satisfies the conditon (*). Since $n(\xi_s) \leq n(O), r(\xi_s) = n(B(s-1))$. By (10)(g), there is an $i < n(\xi_s)$ with $i \in P(s)$, which contradicts the way of taking s.

The proof is completed.

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